ion, is the difference in modeling philosophy. Some engineers prefer to think of time-domain discretization through mathematical finite differencing; others prefer to model with transmission-line networks. Comfort in the modeling concept is far more likely to lead the modeler to more advanced models, as is illustrated by Dr. Gwarek in his interesting paper [1].

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Indefinite Integrals Useful in the Analysis of Cylindrical Dielectric Resonators

DARKO KAJFEZ

Abstract —**Little-known integrals are listed, useful for the evaluation of stored electric energy in cylindrical regions, such as often appear in the analysis of cylindrical dielectric resonators.**

In the analysis of shielded dielectric resonators, it is often necessary to evaluate the stored electric or magnetic energy within a cylindrical region, such as regions 1, 2, and 3 in Fig. 1. The components of the electric field in region 1 are typically expressed in terms of the function

$$\phi_m(k\rho) = K_m(k\rho) + \alpha I_m(k\rho) \tag{1}$$

where $K_m(k\rho)$ and $I_m(k\rho)$ are the modified Bessel functions of order m, k is the radial wavenumber for the corresponding region, and α is a constant such that the tangential electric field vanishes at $\rho = b$. The boundary conditions at $z = 0$ and $z = L$ are not important in the present consideration. Either of these two surfaces may be covered with a perfect electric conductor or, alternatively, form an interface with a neighboring dielectric region.

When the stored electric energy in region 1 is required, the following indefinite integral is needed:

$$\int \left[\phi_m'^2(k\rho) + \frac{m^2}{k^2\rho^2}\phi_m^2(k\rho) \right] \rho \, d\rho = W(\rho). \tag{2}$$

The solution $W(\rho)$ cannot be found in common mathematical handbooks [1], [2]. Nevertheless, the solution exists as follows:

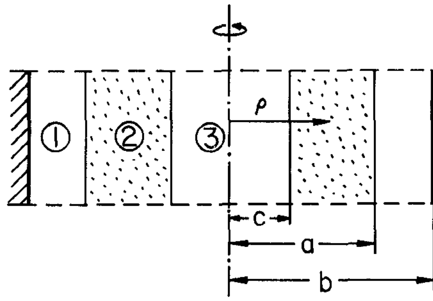

Fig. 1. Cylindrical region filled with inhomogeneous dielectric materials.

$$W(\rho) = \frac{\rho^2}{2}\left[\phi_m'^2(k\rho) + \frac{2}{k\rho}\phi_m(k\rho)\phi_m'(k\rho) - \left(1 + \frac{m^2}{k^2\rho^2}\right)\phi_m^2(k\rho) \right]. \tag{3}$$

When $\alpha = 0$, the result reduces to

$$\int \left[K_m'^2(k\rho) + \frac{m^2}{k^2\rho^2}K_m^2(k\rho) \right] \rho \, d\rho = \frac{\rho^2}{2}\left[K_m'^2(k\rho) + \frac{2}{k\rho}K_m(k\rho)K_m'(k\rho) - \left(1 + \frac{m^2}{k^2\rho^2}\right)K_m^2(k\rho) \right]. \tag{4}$$

The last formula can be found in [3], unfortunately distorted by typographical errors. This formula is useful when radius b of the cylindrical enclosure becomes infinitely large.

The proof of the above formulas consists of taking the derivative of the right-hand side of (3), and showing that

$$\frac{dW(\rho)}{d\rho} = \rho \left[\phi_m'^2(k\rho) + \frac{m^2}{k^2\rho^2}\phi_m^2(k\rho) \right]. \tag{5}$$

The derivation of the above identity is based on the fact that $\Phi_m''(k\rho)$, being a linear combination of modified Bessel functions, satisfies

$$\phi_m''(k\rho) = -\frac{1}{k\rho}\phi_m'(k\rho) + \left(1 + \frac{m^2}{k^2\rho^2}\right)\phi_m(k\rho). \tag{6}$$

Another, similar identity can be obtained for ordinary Bessel functions, needed for evaluation of the stored energy in region 2:

$$\int \left[\psi_m'^2(k\rho) + \frac{m^2}{k^2\rho^2}\psi_m^2(k\rho) \right] \rho \, d\rho = V(\rho) \tag{7}$$

where $\psi_m(k\rho)$ is a linear combination of the ordinary Bessel functions:

$$\psi_m(k\rho) = J_m(k\rho) + \beta Y_m(k\rho). \tag{8}$$

The corresponding solution is

$$V(\rho) = \frac{\rho^2}{2}\left[\psi_m'^2(k\rho) + \frac{2}{k\rho}\psi_m(k\rho)\psi_m'(k\rho) + \left(1 - \frac{m^2}{k^2\rho^2}\right)\psi_m^2(k\rho) \right]. \tag{9}$$

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The author is with the Department of Electrical Engineering, University of Mississippi, University, MS 38677.

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For $\beta = 0$, the last solution reduces to

$$\int \left[J_m'^2(k\rho) + \frac{m^2}{k^2\rho^2} J_m^2(k\rho) \right] \rho \, d\rho$$

$$= \frac{\rho^2}{2} \left[J_m'^2(k\rho) + \frac{2}{k\rho} J_m(k\rho) J_m'(k\rho) + \left(1 - \frac{m^2}{k^2\rho^2}\right) J_m^2(k\rho) \right]. \tag{10}$$

The above formula has been published in [4, p. 58], unfortunately containing typographical errors. The formula is useful for evaluation of energy stored in region 3 or, alternatively, in region 2 when radius c becomes infinitely small.

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